

Remark: Define a map $V \rightarrow F^n$. This map is linear

$$\begin{aligned} \vec{x} &\mapsto [\vec{x}]_{\beta} \\ \text{(HW. } [\underbrace{a\vec{x} + \vec{y}}_V]_{\beta} &= a[\vec{x}]_{\beta} + [\vec{y}]_{\beta}) \end{aligned}$$

Now, suppose V and W are finite-dimensional vector spaces with ordered bases $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ and $\gamma = \{\vec{w}_1, \dots, \vec{w}_m\}$ respectively.
(for V) (for W)

Let $T: V \rightarrow W$ be a linear transformation.

Then for each $1 \leq j \leq n$, $\exists a_{ij} \in F$ ($1 \leq i \leq m$) such that

$$T(\underbrace{\vec{v}_j}_W) = \sum_{i=1}^m a_{ij} \vec{w}_i \quad \text{for } 1 \leq j \leq n,$$

Definition: With this notation as above, we call the matrix

$A \stackrel{\text{def}}{=} (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ the matrix representation
of T in the ordered bases β and γ , and

denoted it as $A = [T]_{\beta}^{\gamma}$.

If $V = W$ and $\beta = \gamma$, then we simply

write ~~$[T]_{\beta}^{\beta}$~~ $[T]_{\beta}$

$$T(\vec{v}_j) = \sum_{i=1}^m a_{ij} \vec{w}_i \quad \text{for } 1 \leq j \leq n,$$

$$T(\vec{v}_j) = \sum_{i=1}^m a_{ij} \vec{w}_i \quad \text{for } 1 \leq j \leq n,$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2n} \\ a_{31} & a_{32} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \vdots & a_{mn} \end{pmatrix}$$

" $[T(\vec{v}_1)]_y$
" $[T(\vec{v}_2)]_y$
" $[T(\vec{v}_n)]_y$

$$T(\vec{v}_1) = \sum_{i=1}^m a_{i1} \vec{w}_i$$

$$\begin{matrix} \uparrow \\ \vec{w} \end{matrix} \Downarrow [T(\vec{v}_1)]_y = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \in F^m$$

$$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \quad \text{for } V$$

$$\gamma = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\} \quad \text{for } W$$

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} | & | & & | \\ [T(\vec{v}_1)]_{\gamma} & [T(\vec{v}_2)]_{\gamma} & \dots & [T(\vec{v}_n)]_{\gamma} \\ | & | & & | \end{pmatrix}$$

$M_{m \times n}$

m

n

Examples:

• Let $A \in M_{m \times n}(F)$. $L_A : F^n \rightarrow F^m$ defined by: $L_A(\vec{x}) \stackrel{\text{def}}{=} A\vec{x}$

Let β and γ be the standard bases for F^n and F^m resp.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots \right\}$$

$$[L_A]_{\beta}^{\gamma} = \left(\begin{array}{c|c|c} \begin{array}{c} | \\ \text{first col. of } A \\ | \end{array} & & \begin{array}{c} | \\ \text{n}^{\text{th}} \text{ col. of } A \\ | \end{array} \\ \hline [A\vec{e}_1]_{\gamma} & \dots & [A\vec{e}_n]_{\gamma} \\ \hline \begin{array}{c} | \\ \vdots \\ | \end{array} & & \begin{array}{c} | \\ \vdots \\ | \end{array} \end{array} \right)$$

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \text{first col. of } A$$

$$= \left(\begin{array}{c|c} \begin{array}{c} | \\ \text{1}^{\text{st}} \text{ col. of } A \\ | \end{array} & \begin{array}{c} | \\ \text{n}^{\text{th}} \text{ col. of } A \\ | \end{array} \\ \hline & \end{array} \right) = A$$

• For $T: P_n(\mathbb{R}) \rightarrow P_{n-1}(\mathbb{R})$ defined as $T(f(x)) = f'(x)$.

Let $\beta = \{1, x, x^2, \dots, x^n\}$ be an ordered basis for $P_n(\mathbb{R})$

Let $\gamma = \{1, x, x^2, \dots, x^{n-1}\}$ be an ordered basis for $P_{n-1}(\mathbb{R})$

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} | & | & & | \\ \textcircled{[T(1)]}_{\gamma} & \textcircled{[T(x)]}_{\gamma} & \dots & \textcircled{[T(x^n)]}_{\gamma} \\ | & | & & | \\ 0 & 1 & & n x^{n-1} \\ | & | & & | \\ & & & \vdots \\ | & | & & | \\ & & & 0 \end{pmatrix} = \begin{pmatrix} | & | & | & | & | \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{pmatrix}$$

Example: $T: M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ defined by:

$$T(A) \stackrel{\text{def}}{=} A^T + 2A$$

↑
transpose

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ — ordered basis}$$

$$T(\beta) = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \right\}$$

$$[T]_{\beta} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Example: $T: P_2(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ defined by:

$$T(f) \stackrel{\text{def}}{=} \begin{pmatrix} f(0) & f(1) \\ 0 & f'(0) \end{pmatrix}$$

Consider ordered basis: $\beta = \{1, x, x^2\}$ for $P_2(\mathbb{R})$

$$\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$T(\beta) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

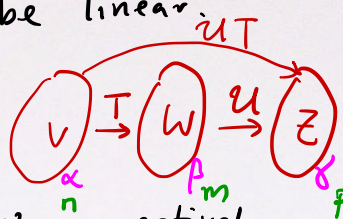
$$\therefore [T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_{4 \times 3}$$

Composition of linear transformations and matrix multiplication

Thm: Let V and W be two vector spaces over the same field F .

And let $T: V \rightarrow W$ and $U: W \rightarrow Z$ be linear.

(i) Then the composition $UT: V \rightarrow Z$ is linear.



(ii) If V, W, Z have ordered bases α, β, γ respectively,

then:

$$\underbrace{[UT]_{\alpha}^{\gamma}}_{M_{p \times n}} = \underbrace{[U]_{\beta}^{\gamma}}_{M_{p \times m}} [T]_{\alpha}^{\beta} \in M_{m \times n}$$

matrix multiplication.

(i) Let $\vec{x}, \vec{y} \in V$ and $a \in F$. Then:

$$U_T(a\vec{x} + \vec{y}) = U(aT(\vec{x}) + T(\vec{y})) = aU_T(\vec{x}) + U_T(\vec{y})$$

$\therefore U_T$ is linear.

(ii) Suppose $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

$$\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$$

$$\gamma = \{\vec{z}_1, \vec{z}_2, \dots, \vec{z}_p\}$$

$$[U]_{\beta}^{\gamma} = \underset{M_{p \times m}(F)}{\overset{\uparrow}{A}} \stackrel{\text{def}}{=} (a_{ik})_{\substack{1 \leq i \leq p \\ 1 \leq k \leq m}}$$

means:

$$U(\vec{w}_k) = \sum_{i=1}^p a_{ik} \vec{z}_i$$

$$1 \leq k \leq m$$

$$= \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{pk} \end{pmatrix}$$

\uparrow
 a_{ik}

$$[T]_{\alpha}^{\beta} = B \stackrel{\text{def}}{=} (b_{kj})_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}} \text{ means } T(\vec{v}_j) = \sum_{k=1}^m b_{kj} \vec{w}_k \text{ for } 1 \leq j \leq n$$

$M^{m \times n}(F)$

$$\begin{aligned} \text{Then: } UT(\vec{v}_j) &= U\left(\sum_{k=1}^m b_{kj} \vec{w}_k\right) \\ &= \sum_{k=1}^m b_{kj} U(\vec{w}_k) \\ &= \sum_{k=1}^m b_{kj} \left(\sum_{i=1}^p a_{ik} \vec{z}_i\right) = \sum_{i=1}^p \left(\sum_{k=1}^m a_{ik} b_{kj}\right) \vec{z}_i \end{aligned}$$

↑
(i, j)-entry of AB

$$\text{So, } [UT]_{\alpha}^{\gamma} = AB = [U]_{\gamma}^{\alpha} [T]_{\alpha}^{\beta}$$

Example: Consider: $U: P_n(\mathbb{R}) \rightarrow P_{n-1}(\mathbb{R})$ defined by:

$$U(f(x)) \stackrel{\text{def}}{=} f'(x)$$

Consider $T: P_{n-1}(\mathbb{R}) \rightarrow P_n(\mathbb{R})$ defined by:

$$T(f(x)) \stackrel{\text{def}}{=} \int_0^x f(t) dt$$

Let α and β be standard ordered bases for $P_n(\mathbb{R})$ and $P_{n-1}(\mathbb{R})$ respectively.

$$[U]_{\alpha}^{\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 & & 0 \\ \vdots & 0 & 2 & 0 & & 0 \\ \vdots & \vdots & 0 & 3 & & \vdots \\ \vdots & \vdots & \vdots & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & & n \end{pmatrix} \text{ and } [T]_{\beta}^{\alpha} = \begin{pmatrix} 0 & 0 & 0 & & & 0 \\ 1 & 0 & 0 & & & 0 \\ 0 & \frac{1}{2} & 0 & & & 0 \\ \vdots & 0 & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & & \vdots \\ & & & & & \frac{1}{n} \end{pmatrix}$$

$$\text{So, } [UT]_{\beta} = [U]_{\alpha}^{\beta} [T]_{\beta}^{\alpha} = \text{Identity matrix} = [I_{P_{n-1}(\mathbb{R})}]_{\beta}$$

Corollary: Let V and W be finite-dimensional vector spaces with ordered basis β and γ respectively.

Let $T: V \rightarrow W$ be linear. Then: for any $\vec{u} \in V$, we have:

$$\underbrace{[T(\vec{u})]_{\gamma}}_{\substack{\uparrow \\ W \\ \downarrow \\ \text{Lin. Transf.}}} = \underbrace{[T]_{\beta}^{\gamma}}_{\text{Matrix multiplication}} \underbrace{[\vec{u}]_{\beta}}_{\downarrow}$$

Proof: Fix $\vec{u} \in V$ and consider two linear transformations:

$$f: \overset{\alpha}{F} \rightarrow \overset{\beta}{V}$$

defined by

$$f(a) = a \vec{u} \in V$$

$$g: \overset{\gamma}{F} \rightarrow W$$

defined by

$$g(a) = a T(\vec{u}) \in W$$

f and g are $\overset{\uparrow}{F}$ linear transformations. Also, $g = T \circ f$.

Let $\alpha = \{1\}$ be the standard ordered basis for F .

$$[T(\vec{u})]_{\gamma} = [g(1)]_{\gamma} = [g]_{\alpha}^{\gamma} = [T]_{\beta}^{\gamma} [f]_{\alpha}^{\beta} = [T]_{\beta}^{\gamma} [f(1)]_{\beta} = [T]_{\beta}^{\gamma} [\vec{u}]_{\beta}$$

"Tof"

$$\begin{array}{ccccc} \alpha & f & \beta & T & \gamma \\ F & \rightarrow & V & \rightarrow & W \end{array}$$

$\underbrace{\hspace{15em}}_{g = T \circ f}$

$$\begin{array}{ccc} \alpha = \{1\} & & \gamma \\ F & \xrightarrow{g} & W \end{array}$$

$$[g]_{\alpha}^{\gamma} = \left(\begin{array}{c} | \\ \cancel{[g(\vec{u}_1)]_{\gamma}} \\ | \end{array} \right) = \left(\begin{array}{c} | \\ [g(1)]_{\gamma} \\ | \end{array} \right)$$

Example: $T: M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ defined by:

$$T(A) \stackrel{\text{def}}{=} A^T + 2A.$$

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$[T]_{\beta} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Let $B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$.

$$[T(B)]_{\beta} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$[B]_{\beta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 5 \\ 4 \\ 0 \end{pmatrix}$$

$$T(B) = \begin{pmatrix} 9 & 5 \\ 4 & 0 \end{pmatrix}$$